

Topological $p_x + ip_y$ Superfluid Phase of a Dipolar Fermi Gas in a 2D Optical Lattice

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In a dipolar Fermi gas, the anisotropic interaction between electric dipoles can be turned into an effectively attractive interaction in the presence of a rotating electric field. We show that the topological $p_x + ip_y$ superfluid phase can be realized in a single-component dipolar Fermi gas trapped in a 2D square optical lattice with this attractive interaction at low temperatures. The $p_x + ip_y$ superfluid state has potential applications for topological quantum computing. We obtain the phase diagram of this system at zero temperature. In the weak-coupling limit, the p-wave superfluid phase is stable for all filling factors. As the interaction strength increases, it is stable close to filling factors $n = 0$ or $n = 1$, and phase separation takes place in between. When the interaction strength is above a threshold, the system is phase separated for any $0 < n < 1$. The transition temperature of the $p_x + ip_y$ superfluid state is estimated and the implication for experiments is discussed.

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Unconventional quantum states of ultra-cold Fermi gases have attracted a lot of attention in the recent theoretical and experimental studies. Among them, one of the most desirable states is the topological superfluid $p_x + ip_y$ phase in a two-dimensional (2D) single-component fermi gas. The zero-energy Majorana modes were predicted to appear inside vertex cores in this p-wave state [1]. Due to their insensibility to local perturbations, Majorana fermions are very appealing for applications in topologically-protected quantum computation [2].

The p-wave superfluid phase was predicted to appear near a p-wave Feshbach resonance in a Fermi gas [3, 4]. However the lifetime of the Fermi gas is severely limited by three-body collisions, and so far this superfluid regime has not been achieved in experiments. The successful creation of $^{40}\text{K}^{87}\text{Rb}$ polar molecules [5] has provided another platform to study unconventional quantum states of Fermi systems. For dipoles aligned parallel to the z -axis, a p-wave superfluid state with the dominant p_z symmetry was predicted in a three-dimensional dipolar fermi gas [6]. In a dipolar fermi gas trapped in $x - y$ plane, a p_x or p_y superfluid state may appear if the angle between the dipole direction and the $x - y$ plane is small enough [7]. The $p_x + ip_y$ superfluid phase was predicted for a 2D dipolar Fermi gas with a circularly-polarized microwave field near the resonant frequency [8]. The d-wave and extended s-wave superfluid phases were also predicted for a similar setup in an optical lattice with a linearly-polarized resonant microwave field [9]. In experiments, however, the precise control of the microwave polarization has been difficult [10].

Here we propose a different method to realize a stable $p_x + ip_y$ topological superfluid phase in a dipolar Fermi gas. For $^{40}\text{K}^{87}\text{Rb}$ or other fermionic polar molecules trapped in a 2D square optical lattice, the dipole direction can be fixed by applying an electric field. If the electric field makes a small angle with the lattice plane and rotates fast around the axis normal to plane, the time-averaged interaction between dipoles is effectively

attractive [11]. The $p_x + ip_y$ topological superfluid phase can be realized in this dipolar Fermi gas at low temperatures due to the attractive interaction.

This work is organized as follows. First we derive the effective Hamiltonian for a single-component dipolar Fermi gas trapped in a square lattice with a fast-rotating electric field. Then we study the $p_x + ip_y$ superfluid state in the self-consistent Hartree-Fock approximation. The phase diagram of this system is obtained at zero temperature. When the interaction is weak, the p-wave superfluid state is stable near filling factors $n = 0$ and $n = 1$, and phase separation occurs in between close to half filling. When the interaction strength is above a critical value, the system is phase separated for any $0 < n < 1$. We also discuss the superfluid transition temperature and its implication for experiments in the end.

Effective Model. We consider a single-component dipolar Fermi gas trapped in a square optical lattice under a rotating electric field

$$\mathbf{E}(t) = E[\cos \varphi \hat{z} + \sin \varphi (\cos \Omega t \hat{x} + \sin \Omega t \hat{y})], \quad (1)$$

where Ω is the rotation frequency, E is the magnitude of the electric field, and φ is the angle between the electric field and \hat{z} direction. The optical lattice is in the $x - y$ plane and the rotation axis is along the \hat{z} direction. In strong electric fields, dipoles are aligned parallel to $\mathbf{E}(t)$. For fast rotations, the effective interaction between dipoles is the time-averaged interaction

$$V_{dd}(r) = \frac{d^2(3 \cos^2 \varphi - 1)}{2r^3}, \quad (2)$$

where d is the dipole moment, and r is the distance between the two dipoles. In the following we consider the case with $\cos \varphi < \sqrt{1/3}$ where the effective interaction is attractive, $V_{dd}(r) < 0$.

The effective Hamiltonian of this dipolar Fermi gas is

given by

$$H = - \sum_{\langle ij \rangle} t(c_i^\dagger c_j + c_j^\dagger c_i) + \frac{1}{2} \sum_{i \neq j} V_{i-j} c_i^\dagger c_j^\dagger c_j c_i, \quad (3)$$

where c_i is the fermion annihilation operator at lattice site $i = (i_x, i_y)$, V_{i-j} is the interaction between fermions at site i and j given by $V_{i-j} = V_{dd}(|\mathbf{r}_i - \mathbf{r}_j|)$, and \mathbf{r}_i is the lattice vector. For simplicity we assume that in the optical lattice the fermion can only hop between nearest neighbors with hopping amplitude t . The interaction strength is the largest for nearest neighbors, denoted by $J \equiv |V_{dd}(a)| = d^2 |3 \cos^2 \varphi - 1| / (2a^3)$, where a is the lattice constant.

The Hamiltonian in Eq. (3) has the important particle-hole symmetry. Under the transformation $c_i^\dagger \rightarrow c_i$ and $c_i \rightarrow c_i^\dagger$, a state with filling factor n turns into another state with filling factor $1 - n$. The chemical potential μ transforms as $\mu \rightarrow -\mu + V(0)$ where $V(0) = \sum_{i \neq 0} V_{0i} = -9.02J$. Note that the hopping amplitude t becomes $-t$ under this transformation, but in the square lattice this phase change can be eliminated by a unitary transformation $c_i \rightarrow -c_i$ for only one sublattice with the same oddity of $i_x + i_y$.

Self-consistent Hartree-Fock approximation. Due to the attractive interaction, fermions tend to pair with each other at low temperatures. In the square lattice, since the interaction strength is the largest between nearest neighbors, and the pairing probability is the largest for fermions from nearest neighbors. Under the constraint of Fermi statistics, the dominant pairing symmetry should be p-wave. To study this superfluid pairing state, we use the self-consistent Hartree-Fock approximation in which the interaction produces the Hartree energy E_h , Fock energy E_x , and the pairing energy E_p given by

$$\begin{aligned} E_h &= \frac{1}{2} \sum_{i \neq j} V_{i-j} n^2, \\ E_x &= -\frac{1}{2} \sum_{i \neq j} V_{i-j} \langle c_i^\dagger c_j \rangle \langle c_j^\dagger c_i \rangle, \\ E_p &= \frac{1}{2} \sum_{i \neq j} V_{i-j} \langle c_i^\dagger c_j^\dagger \rangle \langle c_j c_i \rangle, \end{aligned} \quad (4)$$

where $n = \langle c_i^\dagger c_i \rangle$ is the filling factor. The Hamiltonian in Eq. (3) can be approximated as

$$H' = - \sum_{\langle ij \rangle} t(c_i^\dagger c_j + h.c.) + \frac{1}{2} \sum_{i \neq j} V_{i-j} [n(c_i^\dagger c_i + c_j^\dagger c_j) - (\langle c_i^\dagger c_j \rangle c_j^\dagger c_i + h.c.) + (\langle c_i^\dagger c_j^\dagger \rangle c_j c_i + h.c.)] - E_c,$$

where $E_c = E_h + E_x + E_p$.

In momentum space, we have

$$H' - \mu \hat{N}_f = \sum_{\mathbf{k}} [\xi_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \frac{\Delta_{\mathbf{k}}^*}{2} c_{-\mathbf{k}} c_{\mathbf{k}} + \frac{\Delta_{\mathbf{k}}}{2} c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger] - E_c, \quad (5)$$

where $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} + \Sigma_{\mathbf{k}} - \mu$, $\varepsilon_{\mathbf{k}} = -2t(\cos k_x a + \cos k_y a)$ is the band energy, μ is the chemical potential, $\Sigma_{\mathbf{k}}$ is the Hartree-Fock self-energy given by

$$\Sigma_{\mathbf{k}} = V(0)n - \frac{1}{N} \sum_{\mathbf{k}'} V(\mathbf{k} - \mathbf{k}') n_{\mathbf{k}'},$$

the pairing gap $\Delta_{\mathbf{k}}$ is given by

$$\Delta_{\mathbf{k}} = \frac{1}{N} \sum_{\mathbf{k}'} V(\mathbf{k} - \mathbf{k}') g_{\mathbf{k}'},$$

N is the total number of lattice sites and \hat{N}_f is the fermion number operator,

$$V(\mathbf{k}) = \sum_{m \neq 0} V_m \exp(-i\mathbf{k} \cdot \mathbf{r}_m),$$

$g_{\mathbf{k}} = \langle c_{-\mathbf{k}} c_{\mathbf{k}} \rangle$, and $n_{\mathbf{k}} = \langle c_{\mathbf{k}}^\dagger c_{\mathbf{k}} \rangle$.

The Hamiltonian in Eq. (5) can be diagonalized by Bogoliubov transformation. The Hartree-Fock self-energy and gap can be determined self-consistently

$$\Sigma_{\mathbf{k}} = V(0)n - \frac{1}{N} \sum_{\mathbf{k}'} V(\mathbf{k} - \mathbf{k}') \frac{1}{2} [1 - \frac{\xi_{\mathbf{k}'}}{E_{\mathbf{k}'}} \tanh(\frac{\beta}{2} E_{\mathbf{k}'})], \quad (6)$$

$$\Delta_{\mathbf{k}} = -\frac{1}{N} \sum_{\mathbf{k}'} V(\mathbf{k} - \mathbf{k}') \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}} \tanh(\frac{\beta}{2} E_{\mathbf{k}'}), \quad (7)$$

where the quasi-particle excitation energy $E_{\mathbf{k}}$ is given by $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$ and $\beta = 1/(k_B T)$. The filling factor n can be also determined self-consistently,

$$n = \frac{1}{2} [1 - \frac{1}{N} \sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \tanh(\frac{\beta}{2} E_{\mathbf{k}})]. \quad (8)$$

We numerically solve the Hartree-Fock self-energy equation (6), gap equation (7), and number equation (8) together, and find that the gap $\Delta_{\mathbf{k}}$ has finite magnitude and displays a complex symmetry below a critical temperature. A global unitary transformation can always be applied so that the real part of the order parameter $\Delta_{\mathbf{k}}$

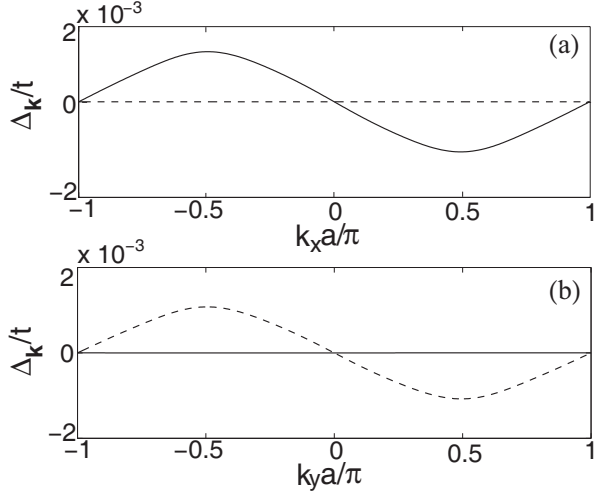


FIG. 1: The gap $\Delta_{\mathbf{k}}$ versus wavevector \mathbf{k} at $J/t = 0.6$, $n = 0.11$, and $T = 0$. The solid and dashed lines are the real and imaginary parts of the gap. (a) At $k_y a/\pi = 0$, the real part of gap has the same sign structure as the function $\sin(k_x a)$, and the imaginary part vanishes. (b) At $k_x a/\pi = 0$, the imaginary part of gap has the same sign structure as the function $\sin(k_y a)$, and the real part vanishes.

remains the same under the transformation $k_y \rightarrow -k_y$ but changes sign when $k_x \rightarrow -k_x$, while the imaginary part of the gap transforms oppositely. As shown in Fig. 1, we find from numerical computation that the real part of the gap has the same sign structure as the function $\sin(k_x a)$ and the imaginary part has the same sign structure as the function $\sin(k_y a)$.

The magnitude of the gap is a monotonically increasing function of the interaction strength J . For the fixed interaction strength, the magnitude of the gap also increases with the filling factor up to half filling, as shown in Fig. 2. Due to the particle-hole symmetry, one state at filling $n = \delta$ is a dual to another state at $n = 1 - \delta$, and they share the same order parameter. Thus the gap of the superfluid state above half filling can always be obtained from the state below half filling. At half-filling, the magnitude of the gap is the largest.

Zero-Temperature Phase Diagram. Due to the attractive nature of the effective dipole-dipole interaction, the free energy of this dipolar Fermi gas is smaller than that of an ideal Fermi gas trapped in the same optical lattice. This energy reduction caused by the interaction increases with the filling factor. When the interaction strength is large enough, the interaction effect is dominant and the system can be unstable. As shown in Fig. 3, at $T = 0$ and $J/t = 0.7$, the chemical potential is a monotonically decreasing function of the filling factor and the compressibility is negative for $0.3 < n < 0.7$, indicating that the $p_x + ip_y$ superfluid state is dynamically unstable. The system is phase separated in the unstable region and be-

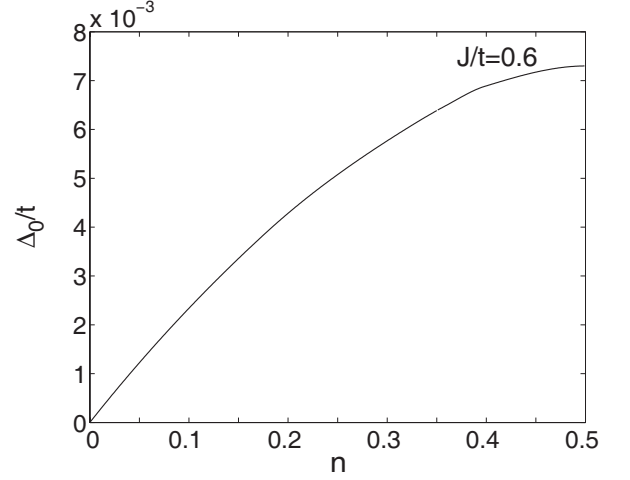


FIG. 2: The gap magnitude Δ_0 versus filling factor n for $J/t = 0.6$ and $T = 0$, where $\Delta_0 = |\Delta_{\mathbf{k}}|$ at $\mathbf{k} = (\pi/2a, \pi/2a)$. At half filling, Δ_0 has the largest value. The gaps for filling factors $0.5 < n < 1$ can be obtained from particle-hole symmetry.

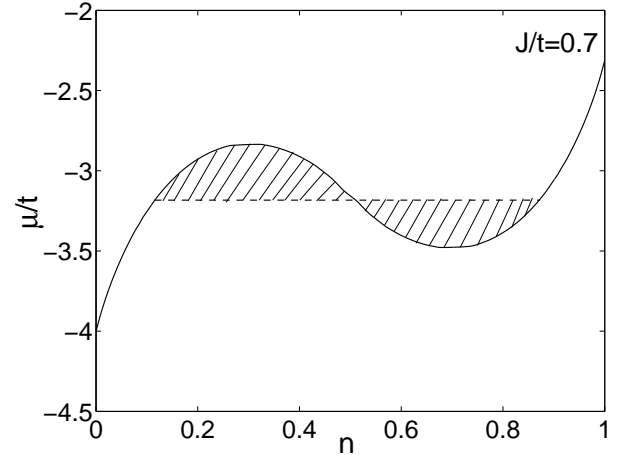


FIG. 3: Chemical potential μ versus the filling factor n at $J/t = 0.7$ and $T = 0$. The dashed line shows the phase-separation region. The shaded areas above and below the dashed line are equal, as required by the phase-separation condition.

comes a mixture of $n = 0.12$ and $n = 0.88$ states, shown by the dashed line in Fig. 3. Due to the particle-hole symmetry, the phase-separation region is symmetric with respect to the half-filling point.

We find that the phase-separation region increases with the interaction strength. In the weak-coupling limit, $J \rightarrow 0$, the phase-separation region vanishes. With strong couplings, for any $0 < n < 1$, the $p_x + ip_y$ superfluid state is unstable and the system becomes a mixture of $n = 0$ and $n = 1$ states which are insulating states due

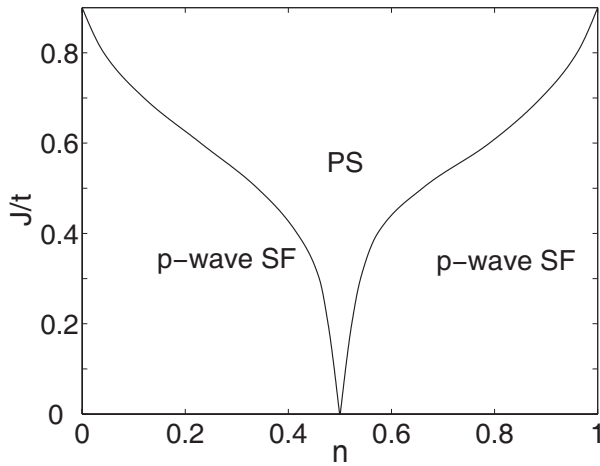


FIG. 4: Phase diagram at zero temperature. The solid line is the boundary between the $p_x + ip_y$ superfluid phase (p-wave SF) and phase-separation region (PS). The superfluid state vanishes when $J > 0.89t$

to zero particle or hole densities. At the critical interaction strength where the superfluid state vanishes, the chemical potential satisfies $\mu(0) = \mu(1)$, where $\mu(n)$ is the chemical potential at filling factor n and $\mu(0) = -4t$. Due to the particle-hole symmetry, $\mu(0) + \mu(1) = V(0)$, the critical interaction strength satisfies $V(0) = -8t$ or $J = 0.89t$. When the interaction strength is above the critical value $J = 0.89t$, the system becomes a mixture of $n = 0$ and $n = 1$ states. The zero-temperature phase diagram of this system is obtained based on these results and shown in Fig. 4.

Discussion and Conclusion. When the temperature increases, the phase-separation region decreases due to the increase in entropy. For example, we find that at $k_B T = 0.1t$ and $J = t$ the phase separation nearly disappears and the superfluid phase exists near half filling. In the superfluid region, as the temperature increases, this 2D dipolar Fermi gas eventually undergoes a Berezinskii-Kosterlitz-Thouless (BKT) transition from the superfluid state to the normal state. At the critical point, the vortex-antivortex pair disassociates and it costs zero free energy to generate a single unbound vortex.

The transition temperature T_{BKT} is given by [14, 15],

$$T_{BKT} = \frac{\pi}{8k_B} \rho, \quad (9)$$

where ρ is the superfluid density. In the $p_x + ip_y$ superfluid state, the superfluid density refers to the geometric mean of eigenvalues of the superfluid density tensor which describes the response of the system to phase twists in the superfluid order parameter. At finite temperatures, the superfluid density is reduced due to excitations of fermionic quasi-particles and bosonic (collective) modes. Here, we assume that the bosonic excitations are negli-

gible and the superfluid density tensor is thus given by

$$\rho_{ij} = \frac{1}{Na^2} \sum_{\mathbf{k}} (n_{\mathbf{k}} \partial_i \partial_j \xi_{\mathbf{k}} - Y_{\mathbf{k}} \partial_i \xi_{\mathbf{k}} \partial_j \xi_{\mathbf{k}}) \quad (10)$$

where $\partial_i f \equiv \partial f / \partial k_i$ and $Y_{\mathbf{k}} = \beta \text{sech}^2(\beta E_{\mathbf{k}}/2)/4$ is the Yoshida distribution. Since the off-diagonal matrix elements of the superfluid density tensor are small, the superfluid density is approximately given by $\rho \approx (\rho_{xx} + \rho_{yy})/2$. By solving Eq. (9) numerically, we obtain the BKT temperature $k_B T_{BKT} = 0.11t$ for $J = t$ and $n = 0.48$.

In experiments, the hopping amplitude is approximately $t = 100\hbar \text{ Hz}$ [13] for a typical optical lattice with $a = 1\mu\text{m}$. The strength of the effective dipole-dipole interaction can be tuned up to $J = 200\hbar \text{ Hz}$ for singlet rovibrational state of KRb molecules with experimental conditions [5, 12]. The potential strength and lattice constant of the optical lattice can be tuned to increase or decrease the hopping amplitude t and interaction strength J , which can be helpful for observing the $p_x + ip_y$ superfluid state.

In summary, we propose that the topological $p_x + ip_y$ superfluid state can be created in a dipolar Fermi gas trapped in a 2D optical lattice in the presence of a rotating electric field, due to the effectively attractive dipole-dipole interaction. The phase diagram of this system is obtained at zero temperature. The $p_x + ip_y$ superfluid state is stable for any filling factor in the weak-coupling limit. As the interaction strength increases, phase separation occurs near half filling. When the interaction strength is above a threshold, the superfluid state is unstable for any filling factor $0 < n < 1$ and the system becomes a mixture of $n = 0$ and $n = 1$ states. We estimate the BKT transition temperature of the topological $p_x + ip_y$ superfluid state which may be accessible in future experiments. We would like to thank T.-L. Ho for helpful discussions. This work is supported by NSFC under Grant No 10974004.

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